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The Geometry of a Parameter Space of Interacting Particle Systems

Christoph Bandt[1]

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We study a four-parameter family of interacting particle systems containing the basic voter model and contact processes. Two processes in this family are related by duality or thinning if and only if their parameters belong to the same orbit of a certain one-dimensional group of linear mappings. This shows that many duals exist.

KEY WORDS: Interacting particle system; duality; thinning.

1. INTRODUCTION

The behaviour of many stochastic systems depends on parameters. A classical case is the Ising model. Here we study a family of interacting particle systems[8, 4] and show how processes with similar properties can be identified from the structure of parameter space. Such a procedure is known from deterministic dynamical systems where attractive periodic orbits determine the behaviour of the processes as well as the geometry of the parameter set—for instance the Mandelbrot set.[10] Our situation is quite different, however.

The parameter set is a four-dimensional simplex $\mathscr{S}$. The geometry consists of a lamination of $\mathscr{S}$ into segments of lines and hyperbolas and yields information on duality and thinnings of interacting particle systems. Some of our results were earlier found by Sudbury and Lloyd,[16, 17, 15] using their machinery of quantum operators. We follow the more intuitive approach of ref. 2.

Interacting particle systems are defined on a countable set V of vertices or sites, for example $V = \mathbb{Z}^d$. We confine ourselves to the simple case

[1] Institut für Mathematik und Informatik, Arndt-Universität, 17487 Greifswald, Germany.

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where each site x can be either in state $\xi(x)=0$ (empty) or in state $\xi(x)=1$ (occupied by a particle). A configuration of our system is a function $\xi: V \to \{0, 1\}$, or just a subset of V, the set of occupied sites $\{x \mid \xi(x)=1\}$. It is custom to denote this set also by ξ.[8, 9, 4] At time $t=0$ we start with a configuration $\xi_0 = A \subset V$, which will then develop in time, subject to chance. Thus ξ_t, the configuration at time t, is a set-valued random variable. The notation ξ_t^A is used when there is need to specify the starting configuration.

Our Subclass. We study rather special particle systems called *edge processes* in ref. 2. For similar definitions see refs. 4, 6, 16, 17, and 12. We assume that V is the vertex set of a finite or infinite graph $G=(V, E)$. Each edge is a set of two vertices, $\{x, y\} = \mathbf{e} \in E$. The most common case is $V=\mathbb{Z}^d$ with edges between nearest neighbours. We require that G has more than one vertex and is locally finite: for some constant K, *no vertex is contained in more than K edges*. We also assume that *G is connected* (otherwise, each component of the graph can be studied separately).

Changes of configurations can always be done at both vertices of a chosen edge $\mathbf{e}=\{x, y\} \in E$, according to the transition rules in Table 1. The process has parameters $a, c, d, e, g \geqslant 0$ with $a+c+d+e+g=1$ which represent the transition rules given in Table 1. Suppose first $(\xi(x), \xi(y))$ is either $(0, 1)$ or $(1, 0)$. Then rule d, the transition to $(0, 0)$, applies to (x, y) (or (y, x), respectively) in a time span of length t with probability

$$dt + o(t) \qquad \text{where} \quad o(t)/t \to 0 \qquad \text{for} \quad t \to 0$$

The probability for rule e is $et + o(t)$, similarly for g. In case $\xi(x)=\xi(y)=0$ nothing will happen since no rule is applicable. Finally, let $(\xi(x), \xi(y)) = (1, 1)$. In a time span of length t, a transition to $(0, 1)$ will occur with probability $ct + o(t)$. The same holds for transition to $(1, 0)$. The probability for

Table 1. Basic Rules[a]

Name	Name	Action	Rule
a	Annihilation	remove both particles	$11 \mapsto 00$
c	Coalescence	remove one particle	$11 \mapsto 01$
d	Dying out	remove particle	$01 \mapsto 00$
e	Exclusion process	shift particle	$01 \mapsto 10$
g	Growth model	add particle	$01 \mapsto 11$

[a] Spontaneous birth $00 \mapsto 01$ or twin birth $00 \mapsto 11$ will not be considered since processes involving these rules have no duals.[8, 4]

transition to $(0, 0)$ is $2at + o(t)$ since both orientations of the edge are applicable and lead to the same result. This convention agrees with the definitions for directed graphs in refs. 16 and 2 but differs slightly from refs. 17 and 15, resulting in differences by the factor 2 in a few statements.

Examples. The voter model has $d = g = 1/2$, and the coalescing random walk $c = e = 1/2$. Contact processes on regular graphs are characterized by $g > 0$ (rate of infection), $c = d$ (recovery irrespective of the state of neighbours) and $a = e = 0$. Many other examples were treated in the literature.[8, 4, 16, 17, 15, 2] Salzano and Schonmann[12] studied contact processes on arbitrary graphs. See ref. 14 for related results and references.

Summarizing we can say that an edge process consists of two ingredients: a graph $G = (V, E)$, and a probability vector (a, c, d, e, g) describing a certain mixture of the rules in Table 1. At each instance an edge is called and a transition of Table 1 performed.

The separation of basic space and transition rule seems to be an advantage of our approach. We are going to study connections between edge processes which are so general that they hold on arbitrary graphs G. Thus, for our purposes, an edge process is represented by a probability vector (a, c, d, e, g). The parameter space of all edge processes is then the four-dimensional simplex

$$\mathcal{S} = \{(a, c, d, e, g) \mid a, c, d, e, g \geqslant 0, a + c + d + e + g = 1\} \tag{1}$$

Results. We consider two processes ξ_t, η_t with parameters (a, c, d, e, g) and (a', c', d', e', g'), respectively. Let $\alpha \neq 1$ denote a real number. ξ_t and η_t are said to be α-duals if the following expectations are equal.

$$\mathbf{E}(\alpha^{|\xi_t^A \cap B|}) = \mathbf{E}(\alpha^{|A \cap \eta_t^B|}) \qquad \text{for} \quad t \geqslant 0 \quad \text{and finite} \quad A, B \subseteq V \tag{2}$$

This definition will be clarified in Section 3, where we also study the case $\alpha = 1$. Here it is enough to know that *duality* is an important tool for particle systems so that we are interested in determining all pairs of α-duals with parameters in $\mathcal{S}$. The best-known cases are $\alpha = 0$ and -1,[6, 8] other values α were first studied by Holley and Stroock.[7]

Following Sudbury and Lloyd,[17] we shall show here that most dual pairs of particle systems are connected by a still simpler relation—one is a *thinning* of the other. This concept depends on a number p for which there are no special values, and so we may conclude that also in the case of duality, no particular α is a priori better or worse than any other value (except $\alpha = 1$, see Theorem 1.3). For $0 < p < 1$, the p-thinning $U^p(\xi)$ of a configuration ξ is obtained by removing each particle of ξ with probability $1-p$, independently of all the others. The process η_t is called a p-thinning of $\xi(t)$ if

$$\eta_0 = U^p(\xi_0) \quad \text{implies} \quad \eta_t = U^p(\xi_t) \qquad \text{for all} \quad t \geqslant 0 \tag{3}$$

Roughly speaking, both processes have the same qualitative behaviour. Everything what can happen to (ξ_t) will also happen to (η_t), for a suitably modified initial configuration. This will be made precise in Section 6 where we show that for initial configurations consisting of one particle, the distribution of the number of particles $|\eta_t|$ can be determined from the distribution of $|\xi_t|$ by a simple formula.

Thinnings of configurations and, more general, of point processes were defined in the seventies. Arratia[1] then found that the annihilating random walk is the 1/2-thinning of the coalescing random walk. However, he did not understand thinning of processes as commutativity of operators (below we rewrite (3) as $U^p S_t = T_t U^p$). Arratia started both processes with the "all 1"-configuration and had to prove rather complicated limit theorems (ref. 1, pp. 920–936).

In ref. 2 it was shown that ξ_t and η_t are duals with $\alpha = 0$ if and only if their parameters are related by an affine reflection R_0 on $\mathbb{R}^5$, that is, $R_0(a, c, d, e, g) = (a', c', d', e', g')$. A simple formula for R_0 and conditions for the existence of 0-duals and for self-duality were derived. Sudbury and Lloyd[16, 17] found similar formulas for arbitrary α in an algebraic way. Only recently Sudbury[15] took the duality formulas to show that there are many more duals than expected before. Approaching the subject from a geometric point of view, we now obtain a complete classification of duals.

First, we divide $\mathscr{S}$ into two-dimensional slices. We define

$$\mathscr{S}_{uv} = \{(a, c, d, e, g) \in \mathscr{S} \mid d+e = u,\ e+g = v\} \qquad \text{for} \quad 0 \leqslant u, v \leqslant 1 \tag{4}$$

Theorem 1.1 (Invariance of slices). If η_t is the α-dual or p-thinning of ξ_t, for some $\alpha \in \mathbb{R}$ or $p \in (0, 1)$, then ξ_t and η_t belong to the same slice $\mathscr{S}_{uv}$.

This is proved in Sections 4 and 5. It is not hard to see that each $\mathscr{S}_{uv}$ is either a quadrilateral, a triangle, an interval or a point (cf. Fig. 2). Nevertheless, it is convenient to study instead of $\mathscr{S}_{uv}$ the whole plane

$$\mathscr{E}_{uv} = \{(a, c, d, e, g) \in \mathbb{R}^5 \mid a+c+d+e+g = 1,\ d+e = u,\ e+g = v\} \supset \mathscr{S}_{uv}$$

and to introduce in $\mathscr{E}_{uv}$ an x-y-coordinate system which we call natural coordinates (see Section 5). $\mathscr{S}_{uv}$ will be within the upper half-plane $\{y \geqslant 0\}$.

Let (x, y) and (x', y') denote the natural coordinates of ξ_t and η_t, respectively.

Theorem 1.2 (Duality formula). η_t is the α-dual of ξ_t if and only if

$$x' = (\alpha - 1)\, y, \qquad y' = x/(\alpha - 1) \tag{5}$$

and $x' = x = 0$ for $\alpha = 1$.

The proof is given in Section 5. Here we discuss some corollaries. First, ξ_t *is* α*-self-dual if and only if* $x = (\alpha - 1)\, y$. Thus for each α, we have a line of self-duality through the origin, as shown in Fig. 1a. For $\alpha = 1$ we have the y-axis, for $\alpha = 0$ the line $y = -x$, and so on. All processes, except those on the x-axis, are self-dual for a unique parameter, $\alpha - 1 = x/y$.

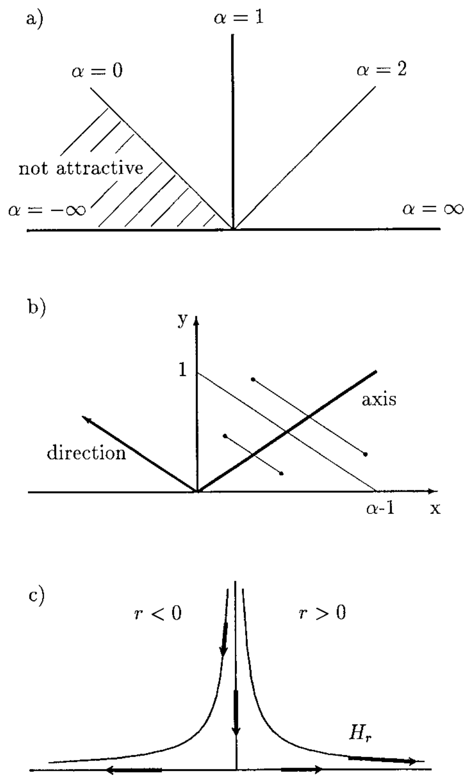

Fig. 1. (a) The lines of self-duality. (b) The action of affine reflections. (c) The trajectories of duality and thinning.

The action of the mapping $R_\alpha(x, y) = ((\alpha-1)\, y, x/(\alpha-1))$ is illustrated in Fig. 1b. R_α is an *affine reflection*, where the axis is given by the vector $(\alpha-1, 1)$, and the direction of reflection by $(1-\alpha, 1)$. For $\alpha \to 1$ both axis and direction approach the y-axis, and for $\alpha \to \pm\infty$ both axis and direction tend to the x-axis—in the limit no reflection is possible. Moreover, each R_α maps the positive part of the y-axis onto the x-axis, either to the left or to the right. Since our processes are all in the upper half-plane $y \geqslant 0$, we have to distinguish two cases. The reflections R_α with $\alpha < 1$ operate on the left of the y-axis, and R_α with $\alpha > 1$ applies to the right-hand part, $\{x \geqslant 0\}$.

For $r \neq 0$ let H_r denote the hyperbola $H_r = \{(x, y) \mid y > 0, xy = r\}$ in $\mathscr{E}_{uv}$. Equation (5) implies that each H_r is preserved by the mappings R_α. Thus all duals of a process with $x \neq 0$, $y > 0$ lie on the hyperbola H_r with $r = xy$. Conversely, if (x_1, y_1) and (x_2, y_2) are two points on H_r, we can easily find a unique reflection R_α which interchanges these two points. Just take the self-duality index of the midpoint, $\alpha - 1 = (x_1 + x_2)/(y_1 + y_2)$. Substituting $x_i = r/y_i$ we see $\alpha - 1 = r/(y_1 y_2) = x_1/y_2 = x_2/y_1$. Thus (5) interchanges the given points as desired, and α is expressed by their self-duality indices as $(\alpha-1)^2 = (x_1 x_2)/(y_1 y_2) = (\alpha_1 - 1)(\alpha_2 - 1)$.[15] Moreover, we proved

Theorem 1.3 (Three types of duality). Two processes are dual if and only if their parameter points fulfil one of the following conditions.

(i) Both points are on the same hyperbola H_r.

(ii) Both points are on the y-axis, and $\alpha = 1$.

(iii) One point is on the y-axis and the other one on the x-axis.

Figure 1c shows hyperbolas and half-axes which contain pairs of duals. Obviously, there are plenty of duals. The process repesenting the origin is singular in the sense that it is self-dual for all α and not dual to anything else.

It turns out that only case (iii) is duality among qualitatively different processes. This includes the classical case of voter model and coalescing random walk (see Fig. 2b). Duality of type (i) and (ii) rather indicates similarity of the corresponding processes, as is stated now and proved in Section 5.

Theorem 1.4 (Trajectories of thinning). Let $0 < p < 1$. The process η_t is the p-thinning of ξ_t if and only if

$$x' = x/p, \qquad y' = py \tag{6}$$

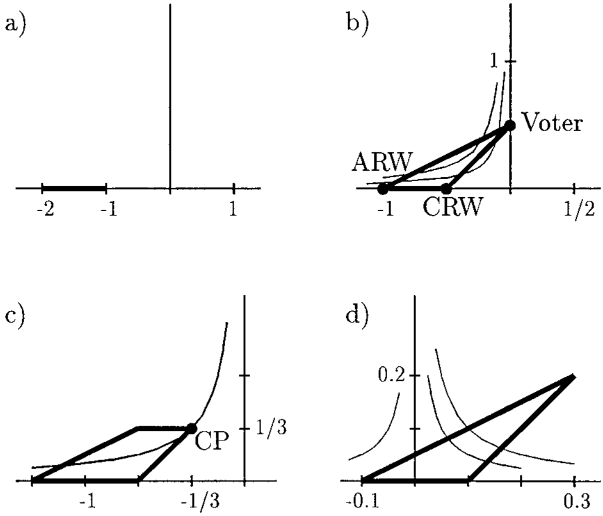

Fig. 2. Concrete examples. (a) $u=v=0$. No duals, $\mathscr{S}_{uv}$ is the interval $a+c=1$ on the x-axis. (b) $u=v=1/2$. This contains the voter model and its classical duals, coalescing and annihilating random walk. (c) $u=v=1/3$. Here $\mathscr{S}_{uv}$ is a quadrilateral, one vertex is a contact process. All contact processes have thinnings, but are not thinnings themselves. (d) $u=4/5$, $v=1/2$. Here $\mathscr{S}_{uv}$ contains processes with self-duality index >1 and <1.

Thus the trajectories of thinning are the hyperbolas H_r and the half-axes drawn in Fig. 1c. Arrows point into the direction of decreasing p. Since $\mathscr{S}_{uv}$ is bounded (cf. Fig. 2), thinnings of ξ_t for arbitrary small $p>0$ will only be possible if ξ_t is on the y-axis.

Using (6), we can define the thinning operator on the parameter space as $U^p(x, y)=(x/p, py)$. This is a so-called hyperbolic rotation. It is obvious that $U^p=R_\beta \cdot R_\alpha$ whenever $p=(\alpha-1)/(\beta-1)$. *The hyperbolic rotations with* $p>0$ (*either* $p\leqslant 1$ *or* $1/p<1$), *and the reflections* R_α *with* $\alpha>1$ *together form a group which acts on the set* $\{x\geqslant 0, y\geqslant 0\}$. *The orbits of this group are the hyperbolas* H_r *with* $r>0$, *the positive x-axis and the positive y-axis.* A similar remark holds for $x\leqslant 0$ and $\alpha<1$.

2. PARTICLE SYSTEMS AS MARKOV PROCESSES

To study duality, we briefly develop the basic concepts of particle systems in a self-contained and somewhat unconvential manner, avoiding the theory of Feller processes and emphasizing analogies with ordinary Markov chains.

The Probability Space. Consider a discrete time Markov chain with countable state space $X=\{1, 2,...\}$ and time-independent probabilities p_{ij} for transition from state i to j. If we are in state i at time $n-1$, then the state j_n at time n can be determined by a uniformly distributed random number u_n in $[0, 1]$ as follows. $j_n=1$ for $0\leqslant u_n<p_{i1}$ and $j_n=2$ for $p_{i1}\leqslant u_n<p_{i1}+p_{i2}$. More generally, $j_n=k$ for $s\leqslant u_n<s+p_{ik}$, with $s=\sum_{j<k} p_{ij}$. Thus the sequence $(u_n)_{n\geqslant 1}$ of random numbers completely determines the sequence (j_n) of states if the initial state j_0 is given. Let $\Omega=\{\omega=(u_n)_{n\geqslant 1}\}=[0, 1]^{\mathbb{N}}$, and let $\mathbf{P}$ the probability measure on Ω which is the infinite product of Lebesgue measure on $[0, 1]$. This is a *universal probability space* for all discrete-time homogeneous Markov chains. It can be used for computer simulations, too.

When we consider homogeneous Markov chains on the same X in continuous time, rates q_{ij} for $i\neq j$ will replace the probabilities p_{ij}. The probability of a transition from i to j within a time interval of length t now is $q_{ij}t+o(t)$. Assuming $\sum_{i\neq j} q_{ij}\leqslant 1$ for simplicity, we can first select a set of transition times t_n as a Poisson process on the time axis, and then determine the new state j for each t_n as above, with q_{ij} instead of p_{ij} (technical detail: q_{ii} is excluded, and for $u_n>\sum_j q_{ij}$ the transition is not executed). The time t_n-t_{n-1} between the occurence of two transitions is exponentially distributed with a fixed parameter, say $\lambda=1$. Writing $t_0=0$ and $r_n=t_n-t_{n-1}$ for $n\geqslant 1$, the development of the whole process is now described by the initial state j_0 and by the sequence $\omega=(r_n, u_n)_{n\geqslant 1}$ of exponentially distributed $r_n\in[0, \infty)$ and uniformly distributed $u_n\in[0, 1]$. Consequently, $\Omega=([0, \infty)\times[0, 1])^{\infty}$ is a universal probability space for continuous time Markov chains.

Now let us consider edge processes as defined in Section 1. The configuration space is uncountable, $X=\{0, 1\}^V$, and to simplify matters we restrict ourselves here to the countable subspace X_0 of *finite* subsets A of V. The essential problem is that the action is not defined globally, with one holding time r_n, but *locally*. For each edge $\mathbf{e}\in E$, independently of all the others, we have a Poisson process $\{t_n^{\mathbf{e}}\}_{n\geqslant 1}$ of intensity $\lambda^{\mathbf{e}}$ (cf. refs. 4, 6, 8, and 9). That is, $r_1^{\mathbf{e}}=t_1^{\mathbf{e}}$ and $r_n^{\mathbf{e}}=t_n^{\mathbf{e}}-t_{n-1}^{\mathbf{e}}$ for $n=2, 3,...$ are independent random numbers, all exponentially distributed with parameter $\lambda^{\mathbf{e}}$, which determine "when the alarm clock for edge $\mathbf{e}$ will ring again.". Let us assume $\lambda^{\mathbf{e}}=2$ for all edges, and let us include the choice of orientation into the Poisson process, so that for $\mathbf{e}=\{x, y\}$, both (x, y) and (y, x) are called with intensity 1, independently.

For each transition time $t_n^{\mathbf{e}}$, $n\geqslant 1$ we need a corresponding uniformly distributed random number $u_n^{\mathbf{e}}$ in $[0, 1]$ for deciding which rule will be applied (similar to Markov chains, take rule a if $0\leqslant u<a$, take c if

$a \leqslant u < a + c$, take d if $a + c \leqslant u < a + c + d$ etc.). All realizations of the $t_n^{\mathbf{e}}$ and $u_n^{\mathbf{e}}$ together form our basic probability space:

$$\Omega = \{\omega = (r_n^{\mathbf{e}}, u_n^{\mathbf{e}})_{n \geqslant 1, \mathbf{e} \in E}\} = ([0, \infty) \times [0, 1])^{\mathbb{N} \times E}$$

and $\mathbf{P}$ is an infinite product measure of exponential distribution on $[0, \infty)$ and Lebesgue measure on $[0, 1]$. All edge processes from $\mathscr{S}$ are defined on one and the same space $(\Omega, \mathbf{P})$.

A Finiteness Argument. The problem here is that within any time interval, infinitely many edges are called and so it might happen that we "do not know where to begin the operation." We have to verify that the $\xi_t^A(\omega)$ are well-defined, for all $A \subset V$, all $t > 0$ and all $\omega \in \Omega$, with possible exception of a subset of Ω of probability 0. We use an argument (cf. [4, Chap. 2]) which formulates particularly well for edge processes.

Lemma 2.1. Let $E_t = E_t(\omega)$ be the set of all edges which are called within $[0, t]$. For a finite set $A \subset V$, let V_t denote the set of all sites y which are connected with a point of A by a path of edges from E_t. Then the set V_t is finite with probability 1, for all $t > 0$.

Proof. We give a proof for $t < 1/2K$, where K was the upper bound for the degree of an arbitrary vertex in the original graph $G = (V, E)$. Then the lemma holds for $0 \leqslant t \leqslant \tau < 1/2K$. We fix τ and apply it again, to the starting set ξ_τ^A instead of A, to show the validity for $t \in [\tau, 2\tau]$ for the starting set A, and then for $t \in [2\tau, 3\tau]$ etc. (Actually, since ξ_τ^A is a random set, this argument involves infinite sums.)

It will be sufficient to consider the case $A = \{x_0\}$. Since $t > 1 - e^{-t}$ for $t > 0$, we have $p := 2K(1 - e^{-t}) < 1$. We determine the probability for the existence of a path of n *different* edges in E_t which starts at x_0. The probability that one of the $2K$ oriented edges with x_0 is called within $[0, t]$ is at most p. Let x_1 be the other point of that edge. The probability that another edge from x_1 is called within $[0, t]$ is smaller than p, and the combination of both events has probability smaller p^2 because of independence of edges. Continuing this way, we find that the probability of a path of length n is smaller than p^n and the probability of an infinite path is 0. ∎

Corollary 2.2. Edge processes are well-defined and transform finite sets into finite sets, with probability 1.

Note that the lemma proves the corollary even for infinite starting configurations ξ_0, in a local sense. We fix a finite set A as "window" in which we observe ξ_t. Since V_t is finite for $t > 0$, we conclude that $A \cap \xi_t$ is well-defined.

Distributions. For discrete time Markov chains, instead of the "casewise" treatment we consider distributions $p(n) = (p_1(n), p_2(n),...)$ at time n, and we find that $p(n) = p(0)\, P^n$, where P is the transition matrix. If the chain has m states, then P defines a mapping (and hence a discrete time dynamical system) on $\mathbb{R}^m$ and in particular on the m-dimensional simplex of all possible distributions p. For continuous time, the dynamical system is given by linear differential equations $p'(t) = p(t)\, Q$, where Q is the rate matrix and $q_{ii} = -\sum_{j \neq i} q_{ij}$. The solution is $p(t) = p(0)\, e^{Qt}$. The exponential matrix represents the transition matrix for a time span t. It is an interpolation of e^{Qn} which now replaces P^n. A similar dynamical system will now be defined for edge processes.

In all what follows, we fix an enumeration $\{A_1, A_2,...\}$ of our infinite state space of all finite subsets $A \subset V$. A distribution is obtained by choosing numbers $p_i = \mathbf{P}\{\xi = A_i\} \geqslant 0$ for $i = 1, 2,...$ with sum 1. It will be called a *random finite subset of* V and will be written as a formal linear combination

$$\xi = \sum_{i=1}^{\infty} p_i A_i \tag{7}$$

This corresponds to writing a distribution of a Markov chain as $p = \sum p_i b_i$ with basis vectors b_i. The proper terminology would be $\xi = \sum_{i=1}^{\infty} p_i \delta_{A_i}$ since we consider ξ not as a random variable, but as a distribution, a probability measure on the finite subsets of V. Our twofold use of A_i as a set and a basis vector may be misleading but saves a lot of δ's. Below, $\emptyset - A$ is well defined as the signed measure $\delta_{\emptyset} - \delta_A$, and $\emptyset$ is distinct from 0. For an edge process ξ_t^A, we use notation (7) with $p_i = \mathbf{P}(\xi_t = A_i \mid \xi_0 = A)$.

Semigroup and Generator. Let $\mathscr{H}$ be the vector space of all formal linear combinations $\eta = \sum_{i=1}^{\infty} r_i A_i$ where the r_i are real coefficients with $||\eta|| = \sum |r_i| < \infty$. In other words, $\mathscr{H}$ is ℓ_1 with standard base $A_1, A_2,...$ Let us now fix t and define $S_t(A_i) = \xi_t^{A_i}$ for all i. The linear extension of this mapping is a linear operator S_t on $\mathscr{H}$ which is continuous with respect to the ℓ_1-norm since $\|A_i\| = \|\xi_t^{A_i}\| = 1$. This is done for all $t \geqslant 0$, and $S_0 = I$. Thus we have a semigroup of operators on $\mathscr{H}$.

However, the operators S_t themselves were not defined in Section 1, only their behaviour for $t \to 0$. We said that within time t, each oriented edge is called with probability $t+o(t)$. Thus within a bounded region and a very short time span, it suffices to regard only the call of one edge since the combined call of more edges has probability of order t^2 or higher. So let us fix an edge $\mathbf{e}=\{x, y\}$ and let $S_t^{\mathbf{e}}$ denote the part of S_t which comes from the action on edge $\mathbf{e}$. We rename the basis vectors A_i in order to define S_t. Let B denote any set without the points x and y, and $B_x = B\cup\{x\}$, $B_y = B\cup\{y\}$, $B_{xy}=B\cup\{x, y\}$. Then $S_t^{\mathbf{e}}(B)=B$ and

$$S_t^{\mathbf{e}}(B_{xy}) = B_{xy} + 2at(B-B_{xy}) + ct(B_x - B_{xy}) + ct(B_y - B_{xy}) + o(t)$$

according to the definitions above and in Section 1. Moreover,

$$S_t^{\mathbf{e}}(B_x) = B_x + dt(B-B_x) + et(B_y - B_x) + gt(B_{xy} - B_x) + o(t)$$

and similarly for B_y.—Now the generator S of the operator semigroup $(S_t)_{t\geqslant 0}$ replaces the concept of rate matrix. S is defined pointwise, $S(A)$ being the derivative at $t=0$ of the curve $(S_t(A))_{t\geqslant 0}$ in $\mathscr{H}$,

$$S(A) = \lim_{t\to 0} \frac{S_t(A)-A}{t} \qquad \text{for} \quad A\subset V \tag{8}$$

if the curve is differentiable at $S_0(A)=A$, i.e. the limit exists. For the $S_t^{\mathbf{e}}$ this is now obvious:

$$\begin{aligned} S^{\mathbf{e}}(B) &= 0 \\ S^{\mathbf{e}}(B_{xy}) &= 2a(B-B_{xy}) + c(B_x + B_y - 2B_{xy}) \\ S^{\mathbf{e}}(B_x) &= d(B-B_x) + e(B_y - B_x) + g(B_{xy} - B_x) \\ S^{\mathbf{e}}(B_y) &= d(B-B_y) + e(B_x - B_y) + g(B_{xy} - B_y) \end{aligned} \tag{9}$$

Actually, this is an *abstract reformulation of the definition of edge process. $S^{\mathbf{e}}$ is given, and the S_t have to be derived. Our problem is to get nice operators.* We extend $S^{\mathbf{e}}$ linearly on $\mathscr{H}$, that is, $S^{\mathbf{e}}(\eta)=\sum r_i S^{\mathbf{e}}(A_i)$ for $\eta = \sum r_i A_i$. This is a continuous operator on our ℓ_1-space $\mathscr{H}$ since

$$\|S^{\mathbf{e}}(A)\| \leqslant \max\{4(a+c), 2(d+e+g)\} =: w \qquad \text{for each} \quad A\subset V$$

We have seen that for small t and finite A, the call of at most one edge has to be regarded to calculate $S_t(A)$. Thus we can define the generator of $(S_t)_{t\geqslant 0}$ as

$$S(A) = \sum_{\mathbf{e}\in E} S^{\mathbf{e}}(A) \qquad \text{with for} \quad A\subset V \tag{10}$$

Here we can replace E by $E(A)=\{\mathbf{e} \mid \mathbf{e}\cap A\neq\emptyset\}$. If A is finite with $|A|$ elements, $S(A)\in\mathscr{H}$ since

$$\|S(A)\|\leqslant K\cdot|A|\cdot w \tag{11}$$

Unfortunately, for infinite graphs G the operator S does not map $\mathscr{H}$ into itself. For $\xi=\sum p_iA_i\in\mathscr{H}$, (11) only implies $\|S(\xi)\|\leqslant K\cdot|\xi|\cdot w$, where $|\xi|=\sum|p_i|\cdot|A_i|$ denotes the average size of the random set ξ. If $|A_i|$ grows much larger than $1/p_i$ for $i\to\infty$, then $|\xi|$ and $\|S(\xi)\|$ become infinite. If $\mathscr{H}_0$ denotes all $\eta\in\mathscr{H}$ with $|\eta|<\infty$, then $S(\mathscr{H}_0)\subseteq\mathscr{H}$, but S would still not be norm continuous.

Once more we see that we have no global command on the development of the process. Following the remark after Corollary 2.2, we introduce a weaker topology on $\mathscr{H}$, generated by seminorms $|\cdot|_A$ for all finite $A\subseteq V$.

$$|\eta|_A=\sum_{C\subseteq A}\left|\sum_{A_j\cap A=C} r_j\right| \qquad \text{for}\quad \eta=\sum r_iA_i \tag{12}$$

$|\eta|_A$ is the ℓ_1-norm of "η restricted to A." Note that $|\xi|_A=\mathbf{P}\{\xi\cap A\neq\emptyset\}$ for random sets ξ. Let A' denote the union of A and all its neighbour vertices in G. Then it is easy to check that $|S(\eta)|_A\leqslant Kw\cdot|\eta|_{A'}$ which means that S is continuous with respect to the new topology. In connection with duality, we shall come back to this *topology of convergence in distribution* (cf. 3.3 and 3.4).

We now show that the exponential of the generator S exists and coincides with the S_t, in a rather weak sense.

$$e^{tS}=I+tS+\frac{1}{2}t^2S^2+\cdots+\frac{1}{n!}t^nS^n+\cdots \tag{13}$$

Proposition 2.3. For $t\leqslant 1/8K$, the series (13) converges pointwise for each basis vector A, with respect to the ℓ_1-metric in $\mathscr{H}$. The limit is $S_t(A)$.

Proof. We apply (11) repeatedly, making use of the fact that the A_i involved in $S(A)$ fulfil $|A_i|\leqslant|A|+1$.

$$\|S^n(A)\|\leqslant K\,|A|\,w\cdot K(|A|+1)\,w\cdot\;\cdots\;\cdot K(|A|+n-1)\,w$$

Thus $S^n(A)\in\mathscr{H}$. Since $w\leqslant 4$, we have for $t\leqslant 1/8K$

$$\frac{1}{n!}t^n\,\|S^n(A)\|\leqslant 2^{-n}\binom{n-1+|A|}{n}=2^{-n}\binom{n+|A|-1}{|A|-1}$$

which is smaller than $2^{-n}n^{|A|}$ for large n. So $\sum_{n=0}^{\infty} (1/n!)\, t^n \, \|S^n(A)\|$ is dominated by a geometric series, and (13) converges pointwise.

Now there are standard arguments to show that (13) is the Taylor series of S_t.[5, 11] From the continuity of the S_u and (8) it is immediate that $S_u S(A) = (\mathrm{d}/\mathrm{d}u)\, S_u(A) = SS_u(A)$. It remains to prove $(\mathrm{d}^n/\mathrm{d}u^n)\, S_u(A) = S^n S_u(A)$ for $n = 2, 3,...$. If S is continuous, then $(\mathrm{d}/\mathrm{d}u)\, SS_u(A) = S(\mathrm{d}/\mathrm{d}u)\, S_u(A)$ and induction applies. In our approach, we use continuity with respect to the weaker topology given by (12). Since the limit of (13) equals $S_t(A)$ in the weak sense, and it exists strongly, it must be $S_t(A)$. ▮

3. DUAL PROCESSES AND DUAL OPERATORS

Duality is an important tool for particle systems. Two processes are dual if, in a certain sense, the second process is the inverse of the first when we run backwards in time.[6, 8, 4] For a formal definition, we introduce a symmetric function $f(A, B) = f(B, A)$ which assigns to any two finite sets $A, B \subset V$ a real number. For random finite sets $\xi = \sum_{i=1}^{\infty} p_i A_i$ and $\eta = \sum_{j=1}^{\infty} q_j A_j$ which we always assume to be independent, $f(\xi, \eta)$ shall then denote *expectation*. This can be expressed as extension of f to a bilinear form on $\mathscr{H}$.

$$f(\xi, \eta) = \sum_i p_i \sum_j q_j f(A_i, A_j) \tag{14}$$

Definition 3.1. Two edge processes ξ_t and η_t are said to be dual with respect to the function f if

$$f(\xi_t^A, B) = f(A, \eta_t^B) \qquad \text{for all finite sets} \quad A, B \subset V \quad \text{and all} \quad t > 0 \tag{15}$$

Actually, the duality concept for processes is a special case of the ordinary duality of operators K, L in Hilbert space:

$$(Ku, v) = (u, Lv) \qquad \text{for all} \quad u, v \in \mathscr{H} \tag{16}$$

or, equivalently, for all u, v from a basis of $\mathscr{H}$. In our case, the scalar product of ξ and η is given by (14), and the operators are S_t, T_t for fixed t, defined on the basis by $S_t(A) = \xi_t^A$ and $T_t(B) = \eta_t^B$. Moreover, we can replace semigroups by their generators, so that the analogy of (15) and (16) is perfect.

Proposition 3.2. Let $f(A, A_j)$ be bounded in A for each j. Two edge processes are duals of each other if and only if (15) holds for their generators:

$$f(S(A), B) = f(A, T(B)) \qquad \text{for all finite} \quad A, B \subset V$$

Proof. Necessity of the condition follows from (8). To show that generators suffice, we use the exponential series (13). Note that (16) carries over from K, L to K^n, L^n (just plug in $v = Lw$) and to linear combinations of them. The boundedness condition for f and (14) imply that $f(\phi_n, B) \to f(\phi, B)$ when $\|\phi_n - \phi\| \to 0$ so that Proposition 2.3 applies. Once (15) is shown for $t \leqslant t_0$, it holds for all t. ∎

The analogy of (15) and (16) is not perfect, though, since $f(A, A)$ need not be positive and $f(\xi, \eta) = \infty$ will be possible. As in 3.2, we shall only assume $f(\xi, A_j) < \infty$ for $\xi \in \mathscr{H}$. Thus f does not generate a scalar product but rather a *weak convergence*,

$$\xi_n \to \xi \qquad \text{if} \quad f(\xi_n, A_j) \to f(\xi, A_j) \qquad \text{for} \quad j = 1, 2,...$$

The Coalescing Duality Function (Harris[6], cf. ref. 8). Let $f(A, B) = 0$ for $A \cap B = \varnothing$ and $f(A, B) = 1$ otherwise. For random sets this describes the convergence defined in (12):

$$\mathbf{P}\{\xi_n \cap A \neq \varnothing\} \to \mathbf{P}\{\xi \cap A \neq \varnothing\} \qquad \text{for each finite} \quad A \subset V$$

If we replace f by $f_0(A, B) = 1$ for $A \cap B = \varnothing$ and $f_0(A, B) = 0$ otherwise, we obtain the same convergence since $f_0(\xi, \eta) = \mathbf{P}\{\xi \cap \eta = \varnothing\}$.

A Family of Duality Functions. For every real $\alpha \neq 1$, we define $f_\alpha(A, B) = \alpha^m$ where $m = |A \cap B|$ denotes the number of common points. $\alpha = 0$ was treated above. For random sets f_α is the expectation

$$f_\alpha(\xi, \eta) = \mathbf{E}(\alpha^{|\xi \cap \eta|})$$

For $\alpha = 1$, we define $f_1(A, B) = |A \cap B|$ which will be justified in Section 4.

Definition 3.3. For any random set ξ and $C \subseteq A \subseteq V$, let

$$q_{C,A}(\xi) = \mathbf{P}\{\xi \cap A = C\}$$

A sequence (ξ_n) of random finite sets is said to converge to ξ in distribution (or weakly, cf. ref. 12) if $q_{C,A}(\xi_n) \to q_{C,A}(\xi)$ for all finite $C \subseteq A \subset V$.

In the sequel we shall sometimes drop arguments like A and ξ when they are either fixed or arbitrary. We recall that convergence $\xi_n \to \xi$ with respect to f_α means $M_A(\xi_n) \to M_A(\xi)$ where

$$M_A(\xi) = \mathbf{E}(\alpha^{|\xi \cap A|}) = \sum_{C \subseteq A} q_C \alpha^{|C|} \tag{17}$$

Thus convergence in distribution implies f_α-convergence. We prove the converse.

Theorem 3.4. For all $\alpha \neq 1$, convergence of random finite sets with respect to f_α is convergence in distribution.

Proof. Let $p_B(\xi) = q_{B,B}(\xi) = \mathbf{P}\{\xi \supseteq B\}$. By the combinatorical principle of inclusion and exclusion, the q_C can be expressed as linear combinations of the p_B.

$$q_C = p_C - \sum_{x \in A \setminus C} p_{C \cup \{x\}} + \sum_{x,\, y \in A \setminus C} p_{C \cup \{x,\, y\}} - \cdots = \sum_{B \supseteq C,\, B \subseteq A} p_B (-1)^{|B \setminus C|} \tag{18}$$

Thus for convergence in distribution it is necessary and sufficient that $p_B(\xi_n) \to p_B(\xi)$ for finite $B \subset V$.

Now let us assume that $M_A(\xi_n) \to M_A(\xi)$ for all $A \subset V$. We want to show that $p_A(\xi_n) \to p_A(\xi)$ for all A. We use induction on $|A|$. For $|A| = 1$, the special form of (17)

$$M_A = \alpha p_A + 1 - p_A$$

shows that $M_A(\xi_n) \to M_A(\xi)$ implies $p_A(\xi_n) \to p_A(\xi)$.

Next, we suppose that $p_B(\xi_n) \to p_B(\xi)$ was proved for all proper subsets $B \subset A$. From the identity

$$M_A = (\alpha - 1)^{|A|}\, p_A + \sum_{B \subset A} (\alpha - 1)^{|B|}\, p_B \tag{19}$$

we can again conclude that $M_A(\xi_n) \to M_A(\xi)$ implies $p_A(\xi_n) \to p_A(\xi)$.

It remains to prove (19). We substitute (18) in (17) and put $j = |C|$.

$$M_A = \sum_{C \subseteq A} \alpha^{|C|} \sum_{B \supseteq C,\, B \subseteq A} p_B (-1)^{|B \setminus C|} = \sum_{B \subseteq A} p_B \sum_{C \subseteq B} \alpha^{|C|} (-1)^{|B \setminus C|}$$

$$= \sum_{B \subseteq A} p_B \sum_{j=0}^{|B|} \binom{|B|}{j} \alpha^j (-1)^{|B|-j} = \sum_{B \subseteq A} p_B (\alpha - 1)^{|B|} \qquad \blacksquare$$

Remark 3.5. The above proof holds for all elements of $\mathscr{H}$ when they are interpreted as signed measures on $\{A_1, A_2,...\}$, or as corresponding signed measures on the space $X = \{0, 1\}^V$ of all configurations where convergence in distribution is known as weak convergence. This includes our definition (12). With this interpretation in mind, we see that $\mathscr{H}$ with f_α-convergence is not complete; its completion is the space $\mathscr{M}$ of all signed measures on X with weak topology. For f_α with $\alpha \neq 1$ the theorem now allows to study $f_\alpha(\mu, A_j)$ with $\mu \in \mathscr{M}$, cf. ref. 8. This will not be done here. We mention that Theorem 3.4 does not hold for $\alpha = 1$. The two random sets $\xi = \frac{1}{2}(\{x\} + \{y\})$ and $\eta = \frac{1}{2}(\emptyset + \{x, y\})$ fulfil $f_1(\xi, A_j) = f_1(\eta, A_j)$ for all j.

4. THE DUALS OF AN EDGE PROCESS

One important question was not asked so far: when do duals really exist? We show that this is often the case for edge processes, and give explicit formulas for the duals. Let us confine ourselves to duality functions $f(A, B)$ which depend on $A \cap B$ only. With a slightly stronger restriction it will turn out that we need only consider the f_α discussed above. We begin with a necessary condition for the dual.

Theorem 4.1. Let $G = (V, E)$ be a graph, and let the duality function be of the form $f(A, B) = h(A \cap B)$ with $h(\{x\}) \neq h(\emptyset)$ for at least one $x \in V$. If the two edge processes (ξ_t) with parameters a, c, d, e, g and (η_t) with a', c', d', e', g' are dual on G then

$$d + e = d' + e' \qquad \text{and} \qquad e + g = e' + g' \tag{20}$$

Proof. Let us take a single edge $\mathbf{e} = \{x, y\}$ from E and suppose that $k-1$ other edges meet the point x. We apply Proposition 3.2 to starting configurations $A, B \subseteq \{x, y\}$. Let us start with $A = B = \{x\}$ and recall (9) and (10) from Section 2. A first change can occur only at one of the k edges $\{x, z\}$ with $z \neq x$. Rules d, e lead to $\xi_t^A \cap B = \emptyset$ while rule g does not change $\xi_t^A \cap B$. Thus

$$h(S(A) \cap B) = t(d + e)\, k(h(\emptyset) - h(\{x\})), \qquad \text{and similarly}$$

$$h(A \cap T(B)) = t(d' + e')\, k(h(\emptyset) - h(\{x\}))$$

This implies $(d + e - d' - e')(h(\emptyset) - h(\{x\})) = 0$. Assuming that $h(\{x\}) - h(\emptyset) = \kappa(x) \neq 0$ we obtain the first assertion in (20). For $A = \{x\}$ and $B = \{y\}$ we have

$$h(S(A) \cap B) = (e + g)(h(\{y\}) - h(\emptyset)) \qquad \text{and}$$

$$h(A \cap T(B)) = (e' + g')(h(\{x\}) - h(\emptyset))$$

Duality now yields $(e+g)\,\kappa(y)=(e'+g')\,\kappa(x)$. The same calculation with x and y interchanged leads to the conclusion $\kappa(y)=\kappa(x)$ and completes the proof of Theorem 4.1 and the duality part of 1.1. ▮

We have seen that κ is constant, and h is constant on singletons. To derive more conditions, we now continue with $A=\{x, y\}$ and $B=\{x\}$.

$$h(S(A)\cap B)=(2a+c+(k-1)(d+e))(h(\emptyset)-h(\{x\}))$$

$$h(A\cap T(B))=(kd'+(k-1)\,e')(h(\emptyset)-h(\{x\}))+g'(h(\{x, y\})-h(\{x\}))$$

Using duality and $d+e=d'+e'$ we obtain

$$(d'-2a-c)\,\kappa=g'(h(\{x, y\})-h(\{x\}))$$

Since $h(\{x\})$ is constant, this implies that $h(\{x, y\})$ must not depend on x and y, either. Putting

$$\alpha=\frac{h(\{x, y\})-h(\{x\})}{\kappa}=\frac{h(\{x, y\})-h(\{x\})}{h(\{x\})-h(\emptyset)} \tag{21}$$

we conclude

$$2a+c=d'-\alpha g' \qquad \text{and} \qquad d-\alpha g=2a'+c' \tag{22}$$

where the second term comes from interchanging A and B. Finally, we put $A=B=\{x, y\}$. We determine the difference $h(S(A)\cap B)-h(A\cap T(B))$ using (3) since the action of neighbouring edges cancels out by (20).

$$h(S(A)\cap B)-h(A\cap T(B))=2(a-a')(h(\emptyset)-h(\{x, y\}))+(c-c')(h(\{x\})$$
$$+h(\{y\})-2h(\{x, y\}))=0$$

Dividing by κ, this can be written as

$$(1+\alpha)(a'-a)=-\alpha(c'-c) \tag{23}$$

Equations (20), (22), and (23) form a system of 5 linear equations for the unknowns a', c', d', e', g' which has a unique solution in terms of a, c, d, e, g and α, except for $\alpha=1$. We get $d-g=d'-g'$ from (20), subtract this from (22) and obtain $(1-\alpha)\,g'=2a+c-d+g$. With

$$\delta=2a+c-d+\alpha g \tag{24}$$

we calculate

$$\begin{gathered}(1-\alpha)(a'-a)=\alpha\delta, \qquad (1-\alpha)(c'-c)=-(1+\alpha)\,\delta \\ (1-\alpha)(d'-d)=\delta, \qquad (1-\alpha)(e'-e)=-\delta, \qquad (1-\alpha)(g'-g)=\delta\end{gathered} \tag{25}$$

Here we see that $a'+c'+d'+e'+g'=a+c+d+e+g=1$. Thus the condition for the existence of duals of $(a, c, d, e, g) \in \mathscr{S}$ can be obtained directly from (25) as a system of inequalities $a' \geqslant 0, \ldots, g' \geqslant 0$ (cf. ref. 2). We formulate our results in another way.

Theorem 4.2 (Necessary condition for duals). Let $G=(V, E)$ be a graph, and let the duality function be of the form $f(A, B)=h(A \cap B)$ with $h(\{x\}) \neq h(\emptyset)$ for one $x \in V$.

(i) For the existence of dual edge processes it is necessary that $h(\{x\})$ is constant on V and $h(\{x, y\})$ is constant on E.

(ii) If the number α defined in (21) is different from 1, the dual of any edge process with parameters a, c, d, e, g is uniquely determined by (24) and (25), provided it exists.

(iii) For $\alpha=1$, duals can only exist when $\delta=0$, in which case (25) implies (22) and (23). ▮

In this sense duality does not depend on the graph or the particular type of duality function h, except for the parameter α.—Instead of deriving further necessary properties of h, we now better turn to the question of sufficiency. The functions $f_\alpha(A, B)=\alpha^{|A \cap B|}$ for $\alpha \neq 1$ and $f_1(A, B)=|A \cap B|$ have parameter α. We now prove that any two processes from $\mathscr{S}$ which are connected by the equations (22) and (23) for some α are really duals with respect to f_α. Thus it is justified to speak of an α-dual, and duality functions other than f_α will not lead to other duals. It could be true that the f_α are essentially (that is, up to a linear transformation) the only possible duality functions. Under the restriction that h depends only on the size $|A \cap B|$, this was proved by Sudbury and Lloyd.[16]

Theorem 4.3 (Sufficient condition for duals). Let two edge processes with parameters a, c, d, e, g and a', c', d', e', g' from $\mathscr{S}$ be given.

(i) If the equations (25) are fulfilled for some $\alpha \neq 1$ and δ from (24) then the edge processes are duals with respect to f_α on every graph G.

(ii) If $\delta=0$ in (24) for $\alpha=1$, and (25) holds, then the edge processes are duals with respect to f_1 on every graph G.

(iii) A process is self-dual if and only if $\delta = 0$ in (24). Thus each process with $g > 0$ is self-dual for a uniquely determined value of α. Processes with $g = 0$ are self-dual only if $2a + c = d$, and in this case α can take any value.

Proof of (i) and (ii). By 3.2, we have to show duality for the generators S, T, and by (4) it suffices to show

$$f_\alpha(S^{\mathbf{e}}(A), B) = f_\alpha(A, T^{\mathbf{e}}(B)) \qquad \text{for each edge} \quad \mathbf{e} \in E$$

So let us fix $\mathbf{e}$ and check the condition for A, B. If $D = (A \cap B) \backslash \mathbf{e}$ has k elements, we set $b = \alpha^k$. Now we go through the four cases of the above proof again, using (3). If $A = B = \{x\}$ then

$$f_\alpha(S^{\mathbf{e}}(A), B)/b = (d+e)(1-\alpha) \quad \text{or } -(d+e) \text{ for } \alpha = 1$$

$$f_\alpha(A, T^{\mathbf{e}}(B))/b = (d'+e')(1-\alpha) \quad \text{or } -(d'+e') \text{ for } \alpha = 1$$

and equality follows from (25). The same holds for $A = \{x\}$, $B = \{y\}$ where

$$f_\alpha(S^{\mathbf{e}}(A), B)/b = (e+g)(\alpha - 1) \quad \text{or } (e+g) \text{ for } \alpha = 1$$

The case $A = B = \{x, y\}$ uses (23) in a similar way:

$$f_\alpha(S^{\mathbf{e}}(A), B)/b = 2a(1-\alpha^2) + 2c(\alpha - \alpha^2) \quad \text{or } -2(2a+c) \text{ for } \alpha = 1$$

The non-symmetric case $A = \{x, y\}$, $B = \{x\}$ leads to

$$f_\alpha(S^{\mathbf{e}}(A), B)/b = (2a+c)(1-\alpha) \quad \text{or } -(2a+c)$$

$$f_\alpha(A, T^{\mathbf{e}}(B))/b = g'(\alpha^2 - \alpha) + d'(1-\alpha) \quad \text{or } (g'-d')$$

where (22) applies. Interchange of A and B or $\{x\}$ and $\{y\}$ is treated similarly. Finally, note that (iii) is a consequence of (i), (ii), (24) and (25). ∎

Thus we have at least one α-dual for each edge process with $g > 0$, and we shall find more below. Sudbury and Lloyd[17] defined a more general *quasi-duality* by using the function

$$f(A, B) = \alpha^{|A \cap B|} \beta^{|A \backslash B|} \gamma^{|B \backslash A|} \varepsilon^{|V \backslash (A \cup B)|} \tag{26}$$

and gave some interesting applications. We conclude this section by showing that although four parameters are involved, on $\mathscr{S}$ there are very few quasi-duals beside the α-duals.

Theorem 4.4 (Few quasiduals). Let G be a graph and $f(A, B)$ a function for finite subsets $A, B \subseteq V$ such that $f(\{x\}, \emptyset) = f(\{y\}, \emptyset) \neq f(\emptyset, \emptyset)$ for two neighbouring sites x, y. Let κ and λ be defined by

$$f(\{x\}, \emptyset) - f(\emptyset, \emptyset) = \kappa, \qquad f(\{x, y\}, \{x\}) - f(\{x\}, \emptyset) = \lambda\kappa$$

and suppose that an edge process (ξ_t) with parameters a, c, d, e, g has a dual on G with respect to f. Then (ξ_t) is either a branching and coalescing random walk, $a = d = 0$, or a biased voter model with diffusion, $a = c = 0$.

Proof. We use 3.2 with $B = \emptyset$ and hence $T_t(B) = \emptyset$. Thus $T(B) = 0$ in $\mathscr{H}$ and $f(A, T(B)) = 0$ by (7). For $A = \{x\}$,

$$f(S(A), B) - f(A, T(B)) = d \cdot (-\kappa) + g \cdot \lambda\kappa = 0$$

implies $\lambda = d/g \geqslant 0$. For $A = \{x, y\}$,

$$f(S(A), B) - f(A, T(B)) = 2a \cdot (-\lambda\kappa - \kappa) + 2c \cdot (-\lambda\kappa) = 0$$

so $a(\lambda + 1) + c\lambda = 0$. Now just consider the cases $\lambda = 0$ and $\lambda > 0$. ∎

To apply this to (26), note that $\varepsilon = 1$ is necessary on infinite graphs and can be assumed for finite V (dividing f by $\varepsilon^{|V|}$, except for the rather degenerate case $\varepsilon = 0$, cf. ref. 17, Section 5). Then $\kappa = \beta - 1$, so either $\beta = 1$ or $a = d = 0$ or $a = c = 0$. Interchanging A and B we see that either $\gamma = 1$ or $a' = d' = 0$ or $a' = c' = 0$. The α-duals are obtained for $\beta = \gamma = 1$.

5. DUALITY AND THINNING IN PARAMETER SPACE

Now we study the parameters of dual processes in $\mathscr{S}$ and prove Theorems 1.2 and 1.4. We know already that the slice $\mathscr{S}_{uv}$ given by $d + e = u$, $e + g = v$ is invariant under taking duals and can be studied separately.

Natural Coordinates. $\mathscr{E}_{uv}$ is the plane in $\mathbb{R}^5$ which contains $\mathscr{S}_{uv}$. Let $o = (-1 + 2u - v, 2 + v - 3u, u - v, v, 0)$ be the origin point of $\mathscr{E}_{uv}$, and let $e^1 = (-1, 1, 0, 0, 0)$ and $e^2 = (1, -2, 1, -1, 1)$ be the unit vectors in x- and y-direction, respectively. Their length is not 1, and they are not orthogonal (the cosine of their angle is $-3/4$) but in our figures the coordinate system is drawn as usual. The coordinates $z = (a, c, d, e, g)$ in $\mathbb{R}^5$ and (x, y) in $\mathscr{E}_{uv}$ are connected as follows.

$$o + xe^1 + ye^2 = (-1 + 2u - v - x + y, 2 + v - 3u + x - 2y, u - v + y, v - y, y)$$

and in the other direction (eliminating u and v from $a = -1 + 2u - v - x + y,...$)

$$x = -(2a + c - d + g), \qquad y = g, \qquad \delta = -x + (\alpha - 1)\, y \tag{27}$$

where δ was introduced in (24). Theorem 4.3 says that z is self-dual if $\delta = 0$, which means $y = x/(\alpha - 1)$. This gives the lines of self-duality through the origin shown in Fig. 1a. For $\alpha = 1$ we have the y-axis, for $\alpha = 0$ the line $y = -x$. Self-duality index $\alpha \geqslant 0$ is equivalent to $d \geqslant 2a + c$ which says that "an occupied site flips faster when its neighbour is a 0."[15] Thus $\alpha \geqslant 0$ characterizes the *attractive processes* in our family.[6, 8, 4]

Proof of 1.2. If $z' = (a', c', d', e', g') = o + x'e^1 + y'e^2$ is the α-dual of z, we can express (25) in vector form:

$$z' - z = \frac{\delta}{1 - \alpha}\,(\alpha,\ -1 - \alpha,\ 1,\ -1,\ 1) = \delta e^1 + \frac{\delta}{1 - \alpha}\, e^2$$

When we substitute (27) for δ, we obtain $x' = (\alpha - 1)\, y$ and $y' = x/(\alpha - 1)$. This proves 1.2 for $\alpha \neq 1$, and for $\alpha = 1$ we can apply 4.2(ii). ▮

Thinning. For $0 < p < 1$, the *p-thinning* of a set $A \subset V$ is the random subset $U^p(A)$ of A obtained by independently removing each particle of A with probability $q = 1 - p$. More formally,

$$U^p(A) = \sum_{k=0}^{|A|} p^k q^{n-k} \sum_{|B| = k,\, B \subseteq A} B \tag{28}$$

U^p extends to a linear operator on $\mathscr{H}$, with $U^p(\xi) = \sum q_i U^p(A_i)$ for $\xi = \sum q_i A_i$. The concept of thinning of edge processes, mentioned in Section 1, can now be reformulated using the semigroups of processes: (η_t) is a p-thinning of (ξ_t) if $U^p S_t = T_t U^p$ for all $t > 0$.

Sudbury and Lloyd (ref. 17, Theorem 11) proved that when the α-dual of (ξ_t) coincides with the β-dual of (η_t), then (η_t) is the p-thinning of (ξ_t), for $p = (\alpha - 1)/(\beta - 1)$. A key for the proof is $\mathbf{E}(\beta^{|U^p(A)|}) = \mathbf{E}((p\beta + 1 - p)^{|A|}) = \mathbf{E}(\alpha^{|A|})$. We prefer to discuss thinnings without duality.

Proof of 1.4. Let (ξ_t) be the edge process with parameters (a, c, d, e, g) and natural coordinates (x, y) in $\mathscr{S}_{uv}$, and let $0 < p < 1$. To see whether (η_t) with parameters (a', c', d', e', g') is a p-thinning of (ξ_t), we have to check the equation $U^p S_t = T_t U^p$ for $t > 0$. As in the proof of 4.3, it suffices to show

$$U^p S^{\mathbf{e}}(A) = T^{\mathbf{e}} U^p(A) \qquad \text{for each edge } \mathbf{e} \text{ and each finite } A \subset V \tag{29}$$

$S^{\mathbf{e}}$ was defined by (9), $T^{\mathbf{e}}$ is the same with a', c', d', e', g'. It suffices to study $A = B_x$ and $A = B_{xy}$ for $B \subset V \backslash \{x, y\}$. We write $U^p(B) = C$ and $C_x = C \cup \{x\}$, $C_y = C \cup \{y\}$, $C_{xy} = C \cup \{x, y\}$. Note that

$$U^p(B_x) = pC_x + qC, \qquad U^p(B_{xy}) = p^2 C_{xy} + pq(C_x + C_y) + q^2 C$$

The rest is formal calculation

$$\frac{1}{p} U^p S^{\mathbf{e}}(B_x) = (d - gq)\, C - (d + e + gp)\, C_x + (e + gq)\, C_y + gpC_{xy}$$

$$\frac{1}{p} T^{\mathbf{e}} U^p(B_x) = d'C - (d' + e' + g')\, C_x + e'C_y + g'C_{xy}$$

Compare coefficients at C, C_y, C_{xy} to obtain

$$d' = d - gq, \qquad e' = e + gq, \qquad g' = gp \tag{30}$$

$$\frac{1}{p} U^p S^{\mathbf{e}}(B_{xy}) = 2(a(1+q) + cq)\, C + (-2aq + c(1-2q))(C_x + C_y)$$
$$- 2p(a + c)\, C_{xy}$$

$$\frac{1}{p} T^{\mathbf{e}} U^p(B_{xy}) = 2(a'p + d'q)\, C + (c'p - (d' + g')\, q)(C_x + C_y)$$
$$- 2((a' + c')\, p - g'q)\, C_{xy}$$

Here it suffices to compare two coefficients.

$$a'p + d'q = a(1+q) + cq, \qquad (a' + c')\, p - g'q = (a + c)\, p \tag{31}$$

On the other hand, when $y' = yp$ and $x' = x/p$ are rewritten in coordinates $a, c, \ldots$ by means of (27) and (20), we also obtain (30) and (31). This completes the proof. ▮

By some tedious calculations, we proved that for graphs with two or three vertices all operators U fulfilling (29) have the form $\gamma U^p + (1-\gamma)\, O$ where $O(A) = \varnothing$ for all A. Thus no "other types of thinnings" seem to exist. A modification of the above argument also shows that $ST = TS$ is not possible for the generators of different edge processes.

6. SOME CONSEQUENCES OF THINNING

Let us give some details on how thinnings of edge processes are related. Suppose η_t is the p-thinning of ξ_t, and both processes are started with δ_y, a single particle at vertex y of our graph. We fix $t>0$. Let $p_k = \mathbf{P}\{|\xi_t|=k\}$ and $\tilde{p}_k = \mathbf{P}\{|\eta_t|=k\}$ for $k=0, 1,\dots$ The generating functions for the number of particles at time t are $g(z)=\sum_0^\infty p_k z^k$ for ξ_t, and $\tilde{g}(z)=\sum_0^\infty \tilde{p}_k z^k$ for η_t. Writing $q=1-p$, the thinning condition implies the following identity.

$$q + p \cdot \tilde{g}(z) = g(q + p \cdot z) \tag{32}$$

On the left-hand side we have the generating function of the number of particles of η_t when it is started with $p \cdot \delta_y + q \cdot \delta_\varnothing$, the p-thinning of the configuration δ_y. The function on the right hand describes the number of particles in a p-thinning of ξ_t (cf. (28)).

Differentiation of (32) gives $pg'(q+pz)=p\tilde{g}'(z)$. In particular $g'(1)=\tilde{g}'(1)$ which means that the expected number of successors of y is the same in η_t and ξ_t. Moreover, (32) can be evaluated using Taylor series:

$$\tilde{p}_k = \frac{p^{k-1}}{k!}\, g^{(k)}(q) = p^{k-1} \cdot \sum_{m=k}^{\infty} \binom{m}{k} p_m q^{m-k}$$

for $k=1, 2,\dots$ and $\tilde{p}_0 = (g(q)-q)/p$. From the last equation it is easy to see that $\tilde{p}_0(t)$ goes to zero for $t \to \infty$ if and only if $p_0(t)$ does. Now let us summarize.

Theorem 6.1 (Properties of thinned process). If η_t is the p-thinning of ξ_t then the distributions of the number of successors of a single particle at vertex y at any fixed time t are connected by equation (32). In particular, one distribution can be calculated from the other, and the expectations of both distributions coincide. Moreover, η_t dies out for $t \to \infty$ if and only if ξ_t does so. ▮

The above calculations could be done for the sets η_t, ξ_t as well as for their cardinalities. This amounts to solving $S_t = (U^p)^{-1}\, T_t U^p$ which is possible since U^p is an invertible operator on the vector space generated by finite configurations.

The methods of this paper apply to particle systems with more than two possible states $\xi(x)$. When the states are 0, 1, and 2, a thinning can be applied either to the 1's or to the 2's, or we can replace some 2's by 1's. However, the number of possible transitions amounts to 34, so that it seems preferable to study parts of parameter space as in ref. 4. In a similar way, our results could be extended to edge processes where "edges can have three or more points". Recently, Theorem 1.4 was applied to obtain a relation between the Bak–Sneppen evolution model and the contact process.[3]

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